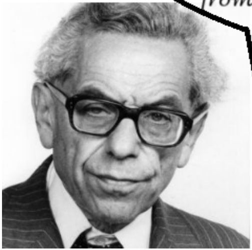
Polowanie na problemy Erdősa

25 VIII 2025

Problems have always been an essential part of my mathematical life. A well chosen problem can isolate an essential difficulty in a particular area, serving as a benchmark against which progress in this area can be measured. An innocent looking problem often gives no hint as to its true nature. It might be like a 'marshmallow,' serving as a tasty tidbit supplying a few moments of fleeting enjoyment. Or it might be like an 'acorn,' requiring deep and subtle new insights from which a mighty oak can develop.



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Here is a list of all the current tags in the system, ordered by the number of problems with that tag. This is a work in progress - if you have suggestions of new tags to be added, or that certain tags should be added or removed from a problem, please let me know.

Sort alphabetically or by total number or by number solved or by number unsolved.

- number theory : 157 solved out of 499
- graph theory : 95 solved out of 232
- ramsey theory : 38 solved out of 96
- geometry : 30 solved out of 91
- additive combinatorics : 36 solved out of 88
- chromatic number : 23 solved out of 52
- unit fractions : 23 solved out of 48
- primes : 10 solved out of 46
- distances : 9 solved out of 42
- combinatorics : 20 solved out of 39
- analysis : 19 solved out of 39
- sidon sets : 5 solved out of 28
- additive basis : 7 solved out of 27
- divisors : 11 solved out of 27
- arithmetic progressions : 9 solved out of 24
- factorials : 7 solved out of 21
- hypergraphs : 7 solved out of 21
- cycles : 8 solved out of 20
- turan number : 6 solved out of 20

★ ★ ERDŐS NEWS ★ ★
VOLUME 1,EDITION 10|MOST WANTED PROBLEMS|1983
WANTED
AGE: 42
ORIGIN:
ADDITIVE
COMBINATORICS
★ PROBLEM SUM I ILOCZYNÓW ★
$250 REWARD
TEKST AKAPITU

Zbiory sum i iloczynów

$$A \subseteq \mathbb{Z}, |A| < +\infty$$

- zbiór sum: $A + A = \{a_1 + a_2 : a_1, a_2 \in A\}$,
- zbiór iloczynów: $AA = \{a_1 a_2 : a_1, a_2 \in A\}$,

Zbiory sum i iloczynów

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$$AA = \{a_1 a_2 : a_1, a_2 \in A\}$$

$$A = \{5, 11, 17, 23\}$$

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Zbiory sum i iloczynów

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$$AA = \{187 \qquad \}$$

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$$A = \{5, 11, 17, 23\}$$

$$A + A = \{28, 34, 10, 16, 22, 40, 46\}$$
$$AA = \{187, 115, 289, 25, 55, 85, 121, 253, 391, 529\}$$

Zbiory sum i iloczynów

$$A + A = \{a_1 + a_2 : a_1, a_2 \in A\}$$
$$AA = \{a_1 a_2 : a_1, a_2 \in A\}$$

$$A = \{5, 11, 17, 23\}, |A| = 4$$

$$A + A = \{10, 16, 22, 28, 34, 40, 46\}$$
$$|A + A| = 7$$
$$AA = \{25, 55, 85, 115, 121, 187, 253, 289, 391, 529\}$$
$$|AA| = 10$$

Zbiory sum i iloczynów

$$A + A = \{a_1 + a_2 : a_1, a_2 \in A\}$$
$$AA = \{a_1 a_2 : a_1, a_2 \in A\}$$

$$B = \{1, 2, 4, 8\}, |B| = 4$$

$$\begin{aligned} B + B &= \{2, 3, 4, 5, 6, 8, 9, 10, 12, 16\} \\ |B + B| &= 10 \\ BB &= \{1, 2, 4, 8, 16, 32, 64\} \\ |BB| &= 7 \end{aligned}$$

Zbiory sum i iloczynów

Jak duży może być zbiór sum?

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$$|A+A| \leqslant \binom{|A|+1}{2} = \frac{|A|^2+|A|}{2}$$

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Moc zbioru sum

Jak duży może być zbiór sum?

$$A = \{a_1, a_2, \ldots, a_n\}, a_1 < a_2 < \ldots < a_n$$

$$a_1 + a_1 < a_1 + a_2 < \ldots < a_1 + a_n < a_2 + a_n < \ldots < a_n + a_n$$

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$$|A + A| \geqslant 2|A| - 1$$

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Moc zbioru sum

Fakt

Jeżeli $A \subseteq \mathbb{Z}$, $|A| < \infty$ i $|A + A| = 2|A| - 1$, to A jest ciągiem arytmetycznym.

Twierdzenie (Freiman)

Jeżeli $A \subseteq \mathbb{Z}$, $|A| < \infty$ i $|A + A| \leqslant 3|A| - 4$, to zbiór A jest zawarty w ciągu arytmetycznym o długości $|A + A| - |A| + 1$.

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Moc zbioru iloczynów

Jak duży może być zbiór iloczynów?

$$2|A| - 1 \leqslant |AA| \leqslant \binom{|A|+1}{2} = \frac{|A|^2 + |A|}{2} \text{ (dla } A \subseteq \mathbb{Z}_{>0}\text{)}$$

- $B = \{1, 2, 4, 8\}, |B| = 4$

$$BB = \{1, 2, 4, 8, 16, 32, 64\}$$
$$|BB| = 7 = 2 \cdot 4 - 1$$

- $A = \{5, 11, 17, 23\}, |A| = 4$

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$$|AA| = 10 = \binom{4+1}{2}$$

Hipoteza Erdősa

Hipoteza (Erdős, 250$)

Niech $\varepsilon > 0$. Wówczas dla dowolnego zbioru skończonego $A \subseteq \mathbb{Z}$

$$\max\{|A+A|, |AA|\} \gg_\varepsilon |A|^{2-\varepsilon}.$$

Hipoteza Erdősa

$$\forall_{A\subseteq\mathbb{Z},|A|<+\infty} \max\{|A+A|,|AA|\} \gg_\varepsilon |A|^{1+c-o(1)}.$$

Co wiemy o c?

- Erdős, Szemerédi, 1983: $c > 0$,
- Nathanson, 1997: $c = \frac{1}{31}$,
- Elekes, 1997: $c = \frac{1}{4}$,
- Solymosi, 2009: $c = \frac{1}{3}$.

Nathanson

2. Sets of small diameter

In this section we obtain a result in the special case of sets of small diameter, and in the next section we show that the main theorem reduces to this special case.

Lemma 1. *Let B be a nonempty, finite set of positive integers such that*

$$\max(B) \leq 2\min(B).$$

Then

$$|E_2(B)| \geq \left(\frac{|B|}{384}\right)^{16/15}.$$

Proof. Let $|B| = k$. If $k < 384$, the inequality is trivial, so we can assume that

$$k \geq 384 = 2^5 12.$$

Then

$$(k/12)^{1/5} \geq 2.$$

Let

$$l = \left[\left(\frac{k}{12}\right)^{1/5}\right].$$

Then

$$l \geq \frac{1}{2}\left(\frac{k}{12}\right)^{1/5} = \left(\frac{k}{384}\right)^{1/5}.$$

Since

$$k \geq 12l^5,$$

it follows that

(1) $$\left[\frac{k}{l}\right] \geq 12l^4.$$

Let $B = \{b_1, \ldots, b_k\}$, where

$$1 \leq b_1 < b_2 < \cdots < b_k \leq 2b_1.$$

For $i = 1, 2, \ldots, [k/l]$, let

$$B_i = \{b_{(i-1)l+1}, b_{(i-1)l+2}, \ldots, b_{il}\} \subseteq B$$

and

$$d_i = b_{il} - b_{(i-1)l+1}.$$

Choose i_0 so that

$$d_{i_0} = \min\{d_i \mid i = 1, \ldots, [k/l]\},$$

and let

$$B^* = B_{i_0}$$

and

$$d^* = d_{i_0}.$$

Suppose that

$$1 \leq i < j \leq \left[\frac{k}{l}\right] \text{ and } j - i \geq 3.$$

If

$$b_1^*, b_2^* \in B^* \text{ and } b_i' \in B_i, b_j' \in B_j$$

then

$$x^* = b_2^* - b_1^* \leq d^*$$

and

$$x = b_j' - b_i' > d_{i+1} + d_{i+2} \cdots + d_{j-1} \geq 2d^* > 0.$$

It follows that

$$b_1^* + b_j' \neq b_2^* + b_i'.$$

Suppose that

$$b_1^* b_j' = b_2^* b_i'.$$

Since $b_i' < b_j'$, it follows that $b_2^* > b_1^*$ and so $x^* > 0$. Since

$$b_j' \leq b_k \leq 2b_1 \leq 2b_1^*,$$

it follows that

$$b_1^* b_j' = b_2^* b_i' = (b_1^* + x^*)(b_j' - x) = b_1^* b_j' + x^* b_j' - x b_1^* - x^* x.$$

and so

$$0 < x^* x = x^* b_j' - x b_1^* \leq b_1^*(2x^* - x) \leq b_1^*(2d^* - x) < 0,$$

which is absurd. Therefore,

$$b_1^* b_j' \neq b_2^* b_i'.$$

It follows that

(2) $$(B^* - B_j) \cap (B^* + B_i) = \emptyset$$

and

(3) $$(B^* \cdot B_j) \cap (B^* \cdot B_i) = \emptyset$$

for every pair i, j of integers such that $j - i \geq 3$.

We shall consider only the sets $B_1, B_4, B_7, \ldots$, that is, the sets B_i such that $i \equiv 1 \pmod 3$. There are at least

$$\frac{1}{3}\left[\frac{k}{l}\right]$$

such sets. Let

$$0 < \theta < 1$$

and

$$\beta = \theta/3 < 1/3.$$

Let

$$E(B^*, B_i) = (B^* + B_i) \cup (B^* \cdot B_i),$$

and let

$$I_1 = \{i \equiv 1 \pmod 3 \mid |E(B^*, B_i)| < l^{1-\beta}\}$$

and

$$I_2 = \{i \equiv 1 \pmod 3 \mid |E(B^*, B_i)| \geq l^{1+\beta}\}.$$

Then

(4) $$|I_1| - |I_2| \geq \frac{1}{3}\left[\frac{k}{l}\right].$$

Suppose that

$$|I_1| \geq \frac{1}{6}\left[\frac{k}{l}\right].$$

Let $i \in I_1$. For $m \in B^* \cdot B_i$, let $\rho(m)$ denote the number of representations of m in the form $b^* b_i'$, where $b^* \in B^*$ and $b_i' \in B_i$. Choose m' such that

$$\rho(m') = \max\{\rho(m) \mid m \in B^* \cdot B_i\}.$$

Since $|B^*| = |B_i| = l$, it follows that

$$l^2 = \sum_{m \in B^* \cdot B_i} \rho(m) \leq \rho(m')|B^* \cdot B_i| < \rho(m') l^{1+\beta},$$

and so

$$\rho(m') > l^{1-\beta}.$$

For $j = 1, \ldots, \rho(m')$, choose $b_j^* \in B^*$ and $b_j' \in B_i$ such that

(5) $$b_j^* b_j' = m'$$

and $b_{j_1}^* \neq b_{j_2}^*$ for $j_1 \neq j_2$. There are $\rho(m')^2$ expressions of the form

$$b_j^* + b_{j'}' \in B^* + B_i,$$

where $j, j' = 1, \ldots, \rho(m')$. Since $i \in I_1$ and $\beta < 1/3$, it follows that

$$\rho(m')^2 > l^{2-2\beta} > l^{1+\beta} > |B^* + B_i'|,$$

and so there exist $b_{j_1}^*, b_{j_2}^* \in B^*$ and $b_{j_3}', b_{j_4}' \in B_i$ such that $b_{j_1}^* \neq b_{j_2}^*$ and

(6) $$b_{j_1}^* + b_{j_3}' = b_{j_2}^* + b_{j_4}'.$$

It follows from (5) that also

(7) $$b_{j_3}^* b_{j_3}' = b_{j_4}^* b_{j_4}'.$$

What we have just shown is that for every $i \in I_1$ there exist four positive integers $b_{j_1}^*, b_{j_2}^*, b_{j_3}^*, b_{j_4}^* \in B^*$ and two positive integers $b_{j_3}', b_{j_4}' \in B_i$ that satisfy equations (6) and (7). However, given any positive integers $b_{j_1}^*, b_{j_2}^*, b_{j_3}^*, b_{j_4}^*$, equations (6) and (7) have at most one solution in integers b_{j_3}', b_{j_4}'. Since the number of quadruples of elements of B^* is exactly l^4, it follows from (1) that if $|I_1| \geq (1/6)[k/l]$, then

$$l^4 \geq |I_1| \geq \frac{1}{6}\left[\frac{k}{l}\right] \geq 2l^4,$$

which is absurd. Therefore,

$$|I_1| < \frac{1}{6}\left[\frac{k}{l}\right],$$

and so, by (4), we have

$$|I_2| \geq \frac{1}{6}\left[\frac{k}{l}\right].$$

Let

$$n \in \bigcup_{i \in I_2} E(B^*, B_i).$$

It follows from (2) and (3) that n belongs to at most two of the sets $E(B^*, B_i)$. Therefore,

$$\begin{aligned} |E_2(B)| &\geq \left|\bigcup_{i \in I_2} E(B^*, B_i)\right| \geq (1/2)\sum_{i \in I_2} |E(B^*, B_i)| \\ &\geq (1/2)|I_2| l^{1+\beta} \geq (1/12)\left[\frac{k}{l}\right] l^{1+\beta} \geq (1/12)(12l^4) l^{1+\beta} \\ &= l^{5+\beta} \geq \left(\frac{k}{384}\right)^{1+\beta/5} = \left(\frac{k}{384}\right)^{1+\theta/15}. \end{aligned}$$

Since this holds for all $\theta < 1$, we obtain

$$|E_2(B)| \geq \left(\frac{k}{384}\right)^{16/15}.$$

This completes the proof of the lemma.

Elekes

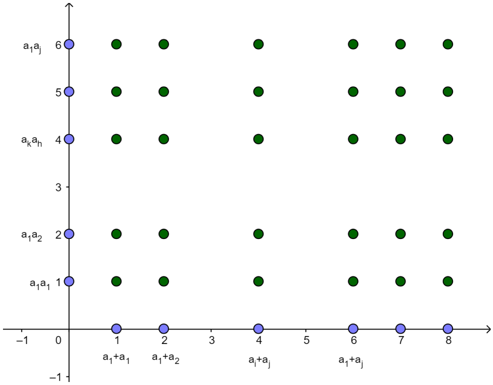
Twierdzenie (Szemerédi, Trotter)

Niech t i N będą dodatnimi liczbami całkowitymi. Ponadto, niech $\mathcal{P}$ będzie zbiorem N różnych punktów na płaszczyźnie, a $l_1, l_2, \ldots, l_M$ będą pewnymi różnymi prostymi. Jeśli każda prosta e_i zawiera co najmniej t punktów z $\mathcal{P}$, to

$$M \leqslant C\frac{N^2}{t^3}.$$

Elekes

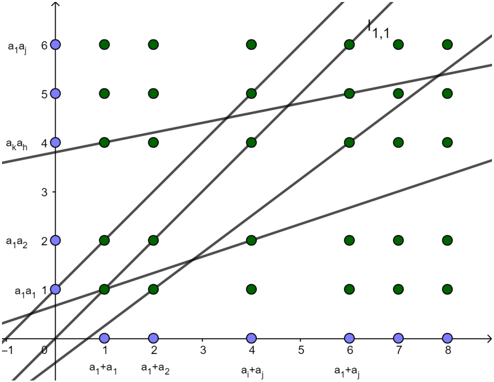
$$M \leqslant C\frac{N^2}{t^3}(M-\text{liczba prostych}, N-\text{liczba punktów}, t-\text{ liczba incydencji})$$



$$\mathcal{P} = (A+A)\times(AA)$$
$$N = |A+A||AA|$$

Elekes

$M \leqslant C\dfrac{N^2}{t^3}$ ($M-$liczba prostych, $N-$liczba punktów, $t-$ liczba incydencji)



$$\forall_{1\leqslant i,j\leqslant |A|} l_{i,j} : y = a_i(x - a_j)$$
$$M = |A|^2$$

Elekes

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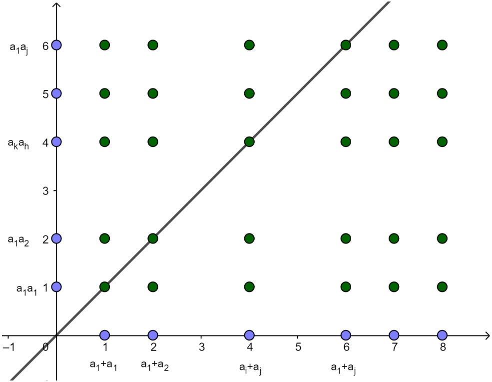
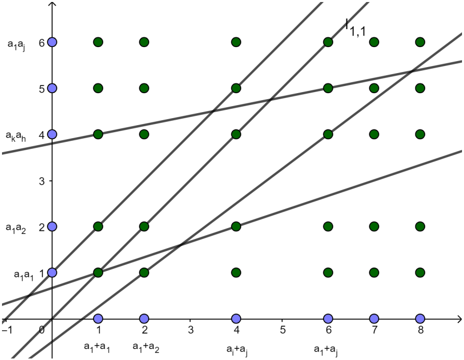


$$\forall_{1\leqslant k\leqslant |A|} a_i(a_k + a_j - a_j) = a_i a_k \Rightarrow (a_k + a_j) \times (a_i a_k) \in l_{i,j}$$

$$t = |A|$$

Elekes

$$M \leqslant C\frac{N^2}{t^3} (M-\text{liczba prostych}, N-\text{liczba punktów}, t-\text{ liczba incydencji})$$



$$N = |A+A||AA|, M = |A|^2, t = |A|$$

$$|A|^2 \leqslant C \cdot \frac{(|A+A||AA|)^2}{|A|^3} \Rightarrow |A+A||AA| \gg |A|^{5/2}$$

Energia multiplikatywna:

$$E(A) = |\{(a, b, c, d) \in A^4 : ab = cd\}|$$

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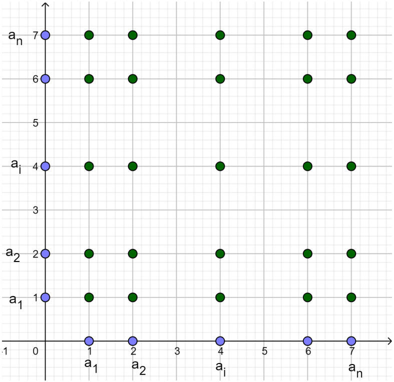
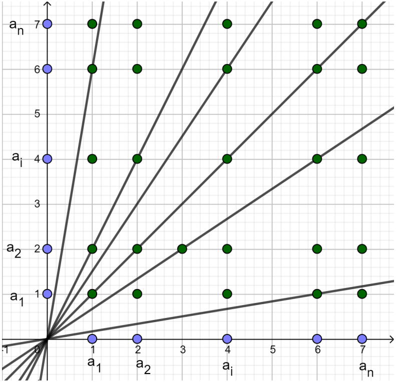
$$\begin{aligned} E(A) &= \sum_{s \in AA} |\{(a,b) \in A^2 : ab = s\}|^2 \\ &\geqslant \frac{1}{|AA|} \left(\sum_{s \in AA} |\{(a,b) \in A^2 : ab = s\}| \right)^2 = \frac{|A|^4}{|AA|} \end{aligned}$$

Załóżmy, że $A \subseteq \mathbb{Z}_{>0}$.
Rozważamy zbiór punktów $(a, b) \in A^2$:



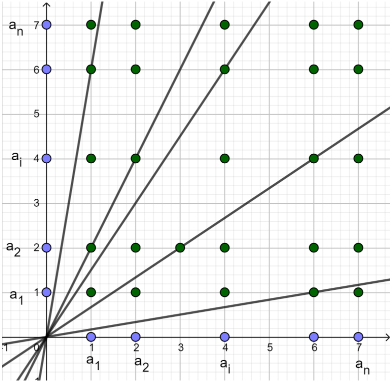
Solymosi

Rozważamy zbiór punktów $(a, b) \in A^2$ i proste $l_s : y = sx$ dla $s \in A/A$:



Solymosi

Dzielimy otrzymane proste na klasy $C_0, C_1, \ldots, C_m$ takie, że $l_i \in C_j \Leftrightarrow 2^j \leqslant |l_i \cap (A \times A)| < 2^{j+1}$ $(m \leqslant \lceil \log |A| \rceil)$.



Solymosi

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$$E(A) = \sum_{s \in A/A} |\{(a,b) \in A^2 : a/b = s\}|^2 = \sum_{s \in A/A} |l_s \cap (A \times A)|^2$$
$$= \sum_{i=0}^{m} \sum_{l_s \in C_i} |l_s \cap (A \times A)|^2$$

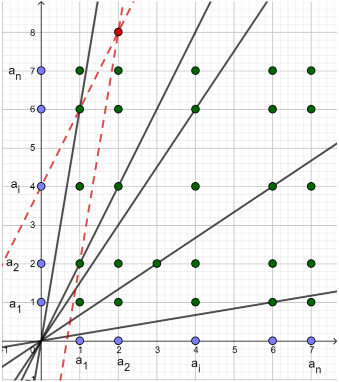
Stąd istnieje indeks K taki, że

$$\frac{E(A)}{\lceil \log |A| \rceil} \leqslant \sum_{l_s \in C_K} |l_s \cap (A \times A)|^2.$$

Załóżmy, że $C_K = \{s_1, s_2, \ldots, s_l\}$ i $s_1 < s_2 < \ldots < s_l$. Wówczas:

$$\frac{E(A)}{\lceil \log |A| \rceil} < l 2^{2K+2}.$$

Załóżmy, że $C_K = \{s_1, s_2, \ldots, s_l\}$ i $s_1 < s_2 < \ldots < s_l$ i rozważmy zbiory $(l_{s_i} \cap A \times A) + (l_{s_{i+1}} \cap A \times A)$:



Zbiory te są parami rozłączne dla różnych parametrów s_i i $|(l_{s_i} \cap A \times A) + (l_{s_{i+1}} \cap A \times A)| = |l_{s_i} \cap A \times A||l_{s_{i+1}} \cap A \times A|$.

Solymosi

$$l \cdot 2^{2K} \leqslant \left| \bigcup_{i=1}^{l} (l_{s_i} \cap A \times A) + (l_{s_{i+1}} \cap A \times A) \right| \leqslant |A + A|^2$$

Solymosi

$$l \cdot 2^{2K} \leqslant \left| \bigcup_{i=1}^{l} (l_{s_i} \cap A \times A) + (l_{s_{i+1}} \cap A \times A) \right| \leqslant |A + A|^2$$

$$\frac{E(A)}{\lceil \log |A| \rceil} < l 2^{2K+2} \Rightarrow \frac{E(A)}{\lceil \log |A| \rceil} \leqslant 4|A + A|^2$$

Solymosi

$$l \cdot 2^{2K} \leqslant \left| \bigcup_{i=1}^{l} (l_{s_i} \cap A \times A) + (l_{s_{i+1}} \cap A \times A) \right| \leqslant |A + A|^2$$

$$\frac{E(A)}{\lceil \log |A| \rceil} < l 2^{2K+2} \Rightarrow \frac{E(A)}{\lceil \log |A| \rceil} \leqslant 4|A + A|^2$$

$$\frac{|A|^4}{|AA|} \leqslant E(A) \Rightarrow \frac{|A|^4}{4\lceil \log |A| \rceil} \leqslant |A + A|^2 |AA|$$

Hipoteza Erdősa

$$\forall_{A\subseteq\mathbb{Z},|A|<+\infty} \max\{|A+A|,|AA|\} \gg_{\varepsilon} |A|^{1+c-o(1)}.$$

Co wiemy o c?

- Erdős, Szemerédi, 1983: $c > 0$,
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- Solymosi, 2009: $c = \frac{1}{3}$,

Hipoteza Erdősa

$$\forall_{A\subseteq\mathbb{Z},|A|<+\infty} \max\{|A+A|,|AA|\} \gg_{\varepsilon} |A|^{1+c-o(1)}.$$

Co wiemy o c?

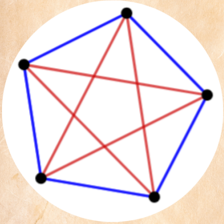
- Erdős, Szemerédi, 1983: $c > 0$,
- Nathanson, 1997: $c = \frac{1}{31}$,
- Elekes, 1997: $c = \frac{1}{4}$,
- Solymosi, 2009: $c = \frac{1}{3}$,
- Bloom, 2025: $c = \frac{1250}{951} \approx 0,33543$.

Problemy pokrewne

★ ★ ERDŐS NEWS ★ ★

VOLUME 1,EDITION 10|MOST WANTED PROBLEMS|1947

WANTED



AGE: 78

ORIGIN: GRAPH THEORY

★ LICZBY RAMSEYA ★

$100 - $1000 REWARD

TEKST AKAPITU

Frank Plumpton Ramsey

Frank Plumpton Ramsey (1903-1930)

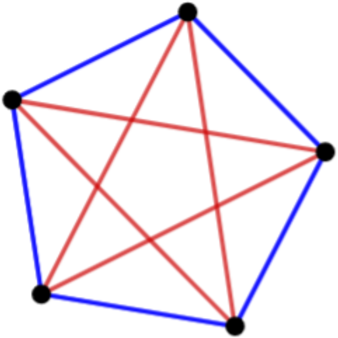
[Verse 1] Born in 1903, learning the world's mystery
Seeing patterns, probability
Yet gripped by his generation's hold
Gazing at the stars above, nothing unfolds

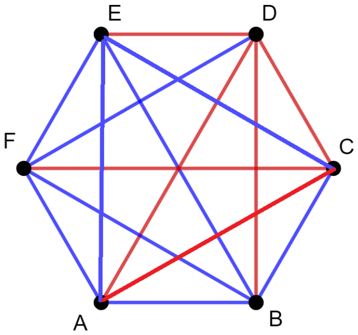
[Verse 2] At 27, walking the path of life
Filled with strife, the end in sight
Each step a mystery, joy and love intertwined
Sent from above, but pain and loss find

[Chorus] Endless stride against the tide
Never bending, always in his stride
Through mystery, pain, and love, he goes
Searching for answers, as the story unfolds

„*A song for Ramsey*", Sicong Meng, Sabrina Nguyen, Gary Tse `https://www.sfu.ca/~vjungic/RamseyProjects/book-1.html`

Liczby Ramseya





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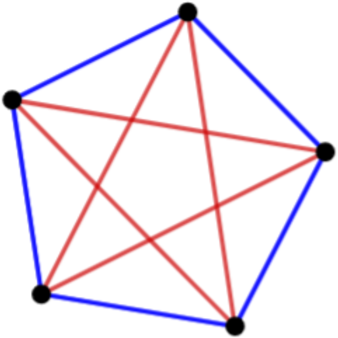

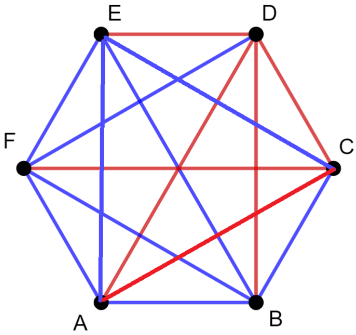

Liczbą Ramseya $R(r, s)$ nazywamy najmniejszą liczbę naturalną n taką, że każde kolorowanie grafu pełnego K_n zawiera czerwony podgraf K_r lub niebieski podgraf K_s.

Liczby Ramseya

Twierdzenie (Twierdzenie Ramseya)

$$\forall_{r,s\geqslant 2} R(r,s) \leqslant R(r-1,s) + R(r,s-1)$$

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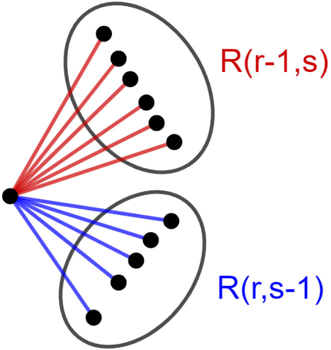

Oszacowanie górne

Twierdzenie (Erdős, Szekeres, 1935)

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$$R(k,k) \leqslant \binom{2k-2}{k-1} \wedge k! \sim k^k e^{-k}\sqrt{2\pi k} \Rightarrow R(k,k) \leqslant c\frac{4^k}{\sqrt{k}}$$

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- $\mathbb{P}(A_S) = 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}}$. Stąd

$$\mathbb{P}\left(\bigcup_S A_S \right) \leqslant \binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1.$$

Problem Erdősa

Problem (Erdős, 100/250/1000$)

Znajdź granicę $\lim_{k\to\infty} R(k,k)^{1/k}$.

$$\sqrt{2} \leqslant \liminf_{k\to\infty} R(k,k)^{1/k} \leqslant \limsup_{k\to\infty} R(k,k)^{1/k} \leqslant 4$$

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- $\mathbb{E}(X) = \sum_S \mathbb{E}(I_S) = \binom{n}{k} 2^{1-\binom{k}{2}}$. Zatem $\exists_\omega X(\omega) \leqslant \binom{n}{k} 2^{1-\binom{k}{2}}$.

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- $\mathbb{E}(X) = \sum_S \mathbb{E}(I_S) = \binom{n}{k} 2^{1-\binom{k}{2}}$. Zatem $\exists_\omega X(\omega) \leqslant \binom{n}{k} 2^{1-\binom{k}{2}}$.
- Dla kolorowania wyznaczonego przez ω usuwamy po jednym wierzchołku z każdego monochromatycznego zbioru S mocy k.

$$R(k,k) \geqslant n - \binom{n}{k} 2^{1-\binom{k}{2}}.$$

Oszacowania dolne

- Erdős, 1947: $R(k,k) \geqslant k \cdot \sqrt{2}^k \left(\frac{1}{e\sqrt{2}} + o(1)\right),$
- Shearer, 1982: $R(k,k) \geqslant k \cdot \sqrt{2}^k \left(\frac{1}{e} + o(1)\right),$
- Spencer, 1975: $R(k,k) \geqslant k \cdot \sqrt{2}^k \left(\frac{\sqrt{2}}{e} + o(1)\right)$

Oszacowania górne

- Erdős, Szekeres, 1935: $R(k,k) \leqslant c\frac{4^k}{\sqrt{k}}$,
- Campos, Griffiths, Morris, Sahasrabudhe, 2023: $R(k,k) \leqslant (4 - \frac{1}{128})^k$,
- Gupta, Ndiaye, Norin, Wei, 2024: $R(k,k) \leqslant (3,7992\ldots)^k$.

Problemy pokrewne

Problem Erdősa

https://www.erdosproblems.com

OPEN - \$10000

Let $r_k(N)$ be the largest possible size of a subset of $\{1, \ldots, N\}$ that does not contain any non-trivial k-term arithmetic progression. Prove an asymptotic formula for $r_k(N)$.

#142: [Er80][Er81][Er97c] additive combinatorics, arithmetic progressions

OPEN - \$500

Does every set of n distinct points in $\mathbb{R}^2$ determine $\gg n/\sqrt{\log n}$ many distinct distances?

#89: [Er57][Er61][Er81][Er82e][Er83c][Er85][Er90][Er92e][Er95][Er97b][Er97c][Er97e][Er97f] geometry, distances

OPEN - \$500

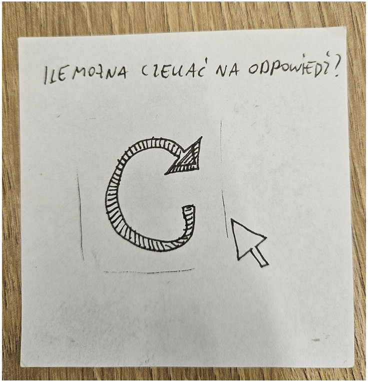
Let the van der Waerden number $W(k)$ be such that whenever $N \geq W(k)$ and $\{1, \ldots, N\}$ is 2-coloured there must exist a monochromatic k-term arithmetic progression. Improve the bounds for $W(k)$ - for example, prove that $W(k)^{1/k} \to \infty$.

#138: [Er57][Er61][Er73][Er74b][Er75b][Er77c][ErGr79][Er80][ErGr80][Er81][Er97c] additive combinatorics

OPEN - \$78

Let $f(n)$ be minimal such that in $(n, n + f(n))$ there exist distinct integers $a_1, \ldots, a_n$ such that $k \mid a_k$ for all $1 \leq k \leq n$. Obtain an asymptotic formula for $f(n)$.

#710: [ErPo80][Er92c] number theory



Agnieszka Cioch, „Znajdź parę”