Asymmetry of Cantorian Mathematics from a categorial standpoint: Is it related to the direction of time?

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Abstract Category theory is symmetric in the sense that all definitions, theorems and proofs have uniquely defined duals, obtained by formal reversing of arrows. In contrast, the products and coproducts in typical categories whose objects are sets endowed with basic algebraic, topological etc. structures of *Cantorian Mathematics* show a specific lack of symmetry.
A philosophical question is raised: What features of mathematics and mathematical thinking are related to this phenomenon? Some hints suggest that this is related to the role of the concept of a function in mathematics and the domination of many-to-one thinking. This in turn may be attributed to implicit thinking in terms "causes precede the effects" and to the arrow of time.

1 Introduction

Initially (since 1945) category theory was regarded as a convenient conceptual language for certain aspects of mathematical theories. Later, however, new ideas developed by Lawvere and others showed that topos theory could provide a unified framework for set theory, logic and a good part of mathematics [12]. Consequently, category theory was viewed as a new contender for a foundation of mathematics along with set theory, or — as Mac Lane would put it — as a proposal for the organization of Mathematics [16, pp. 398–407], [17, p. 331].

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The purpose of this paper[1] is to use categorial concepts to highlight a certain feature of *Cantorian Mathematics*; this term refers here to basic mathematical structures of algebra, topology, functional analysis etc. expressed in terms of set theory, as they were conceived prior to the emergence of category theory, i.e., by the middle of the 20th century. We will consider categories in which objects are sets[2] provided with specific structures, while morphisms are structure preserving maps (homomorphisms, continuous maps etc.).

General category theory is fully symmetric in the sense that each definition, each theorem and each proof has its uniquely defined **dual**, obtained by the process: *reverse all arrows*, that is, by the following replacements in the theory:

- each expression of the type $\alpha\beta = \gamma$ is replaced by $\beta\alpha = \gamma$;
- in each morphism the word "domain" is replaced by "codomain" and "codomain" is replaced by "domain", that is, each $\alpha\colon A \to B$ is replaced by $\alpha\colon B \to A$;
- arrows and composites are reversed, while the logical terms are unchanged [14], [15, pp. 31–33], [23].

Each statement of the theory has a unique dual statement, e.g., the dual of "α is monic" (i.e., a monomorphism) is "α is epic", and vice versa; the dual of "A is an initial object" is "A is a terminal object"; the dual of a product is a coproduct.

In an axiom system for category theory, the dual of each axiom is also an axiom. Consequently, in any proof of a theorem, replacing each statement by its dual gives a valid proof of *the dual theorem*. This is the *duality principle* in category theory.

We will show that — from this point of view — Cantorian mathematics is specifically asymmetric.

2 Products and coproducts

As a crucial example we consider the notion of a product of an indexed family of objects $\{A_t\}_{t\in T}$ in a category $\mathcal{E}$, defined as an object P together with a family of morphisms $\{\pi_t\colon P \to A_t\}_{t\in T}$ (called projections) having the unique factorization property: for every object X and every family of morphisms $\{\xi_t\colon X \to A_t\}_{t\in T}$ there exists a unique morphism $\theta\colon X \to P$ such that the diagrams commute, i.e., $\pi_t\theta = \xi_t$ for $t \in T$. If a product exists, it is unique up to commuting isomorphism.

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[2] It is assumed here that all sets, functions, topological spaces etc. considered here are small sets, [15, p. 22].

A coproduct (also called a categorial sum) of a family of objects $\{A_t\}_{t\in T}$ is defined dually as an object S together with a family of morphisms (called injections) $\{\sigma_t\colon A_t \to S\}_{t\in T}$ such that for every object X and every family of morphisms $\{\xi_t\colon A_t \to X\}_{t\in T}$ there exists a unique morphism $\theta\colon S \to X$ such that $\theta\sigma_t = \xi_t$ for $t \in T$.

2.1 Products in certain categories of sets with structures

Most of the following examples are concrete categories, i.e., categories $\mathcal{E}$ equipped with a faithful functor $U : \mathcal{E} \to \mathbf{Set}$. Symbols of specific categories used here generally follow those of [15, p. 12]. In the category **Set** of (small) sets and functions, the product of $\{A_t\}_{t\in T}$ is the cartesian product $A = \prod_{t\in T} A_t$ with the coordinate projections $\pi_t\colon A \to A_t$ $(t \in T)$.

In the category **Top** of topological spaces and continuous maps, in its full subcategory **Comp** of compact Hausdorff spaces, in the categories $\mathbf{Set}_*$ of sets with selected base-points and base-point preserving functions, in the analogous category $\mathbf{Top}_*$, in the category **Ord** of partially ordered sets and non-decreasing maps, in the categories **Grp**, **Ab** and **AbComp** of groups (resp. abelian groups and compact abelian groups) and their homomorphisms (resp. continuous homomorphisms) — in all these categories the product $\prod_{t\in T} A_t$ is the cartesian product endowed with a suitable structure (in **Ord** it is the cardinal product in the sense of [2, I. §7], [23, 3.3.8]).

The category $\mathbf{Ban}_1$ of Banach spaces and linear contractions, i.e., linear operators of norm $\|T\| \leq 1$ (called also short linear operators), may appear to be an exception to the rule, as the cartesian product of infinitely many Banach spaces is not a Banach space. In fact, their product is the ℓ^∞-product consisting of all $\{x_t\}_{t\in T}$, $x_t \in X_t$, such that $\sup_{t\in T} \|x_t\| < \infty$. However, this exception may be regarded as spurious. If we adjust the concept of the carrier of the Banach space structure, replacing the whole vector space X by its closed unit ball $\{x \in X\colon \|x\| \leq 1\}$ (which actually determines the geometric structure of the whole space) and replacing the category $\mathbf{Ban}_1$ by the category $\mathbf{Ban}_{\bigcirc}$ of closed unit balls and restrictions of linear contractions, then the product object becomes simply the cartesian product of balls.

Another exception is the category **AbTor** of abelian torsion groups. The direct product $A = \prod_{t\in T} A_t$ of such groups need not be a torsion group, e.g., $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots$ However, the torsion-subgroup P of A consisting of all torsion elements (i.e., all elements of finite order) is a product in **AbTor**, [1, 10.20].

2.2 Coproducts in the same categories

In contrast to the preceding, coproducts may be markedly different from each other. This is clearly shown in the following Table 1.

In the categories **Set**, **Top**, $\mathbf{Top}_*$, **Comp** and $\mathbf{Set}_*$ coproducts are based on the same construction, namely on the disjoint union $A = \bigsqcup_{t\in T} A_t$, defined as $\bigcup_{t\in T}(A_t \times \{t\})$, with obvious injections $\sigma_t \colon A_t \to A$. In **Top** the coproduct $\bigsqcup_{t\in T} A_t$ is equipped with the disjoint union topology [5, Ch. 2, §2, §4]; in $\mathbf{Top}_*$ the coproduct is the wedge sum, i.e., the quotient of the disjoint union obtained by identifying the base points to a single point; in **Comp** copoducts are the Stone–Čech compactifications of disjoint unions. In **Ord** it is the cardinal sum [2, I. §7], [23, 3.3.10]).

However, in other categories coproducts may differ basically. In **Grp** the coproduct is the free product of groups, whereas in its full subcategory **Ab** the coproduct is the direct sum $\bigoplus A_t$ (called also the external direct sum); thus the **Ab**-coproduct of two copies of the cyclic group $\mathbb{Z}$ is commutative, whereas their **Grp**-coproduct is not commutative (moreover, its center is trivial).

In **AbComp** the coproduct is the Bohr compactification ([11], [8, p. 430], [9], [21, pp. 249–254]) of the direct sum $\bigoplus A_t$, provided with a coproduct topology [18]). This is particularly interesting in view of the Pontryagin duality. A locally compact abelian group G is compact if and only if its dual group $\widehat{G}$ (the group of characters, i.e., continuous homomorphisms $G \to \mathbb{R}/\mathbb{Z}$) is discrete [13, Ch. VII], [8, §24]; consequently, the category **Ab** (which may be regarded as that of discrete abelian groups) is equivalent (in the sense of [15, IV.4]) to the opposite category (i.e., dual) of **AbComp**. Thus, one might expect a somehow "dual behavior" of their products and coproducts; yet, the products in both categories are akin to Cartesian products and coproducts to direct sums.

In the category $\mathbf{C^*algcom}_1$ of commutative C^*-algebras with units and their homomorphisms, the product is an ℓ^∞-product while the coproduct is the injective tensor product $\widehat{\widehat{\bigotimes}} A_t$ (called also the weak tensor product), [21, pp. 355–361]. By the Gelfand duality theorem, $\mathbf{C^*algcom}_1$ is equivalent to the dual of **Comp** [21, subsections 10.2, 12.6, 13.3], so the situation is analogous to that with products and coproducts in **Ab** and **AbComp**.

In the category **CRng** of commutative rings with units and unit-preserving ring homomorphisms the product of a family $\{R_t\}_{t\in T}$ is — as in any category of algebras of the same type — the Cartesian product $A = \prod A_t$ with suitable operations and coordinate projections, whereas the coproduct of this family is the tensor product of rings $\bigotimes R_t$ (i.e., the tensor product over $\mathbb{Z}$ for rings as $\mathbb{Z}$-algebras, [20, p. 65]).

In the category $\mathbf{Ban}_1$ of Banach spaces and linear contractions the coproduct of a family $\{X_t\}_{t\in T}$ is its ℓ^1-sum consisting of all $\{x_t\}_{t\in T}$ such that $\sum_{t\in T} \|x_t\| < \infty$.

Symbol of the category objects morphisms	**Product** **of a family** $\{A_t\}_{t\in T}$ **of objects**	**Coproduct** **of a family** $\{A_t\}_{t\in T}$ of objects
Set sets functions	Cartesian product $\prod_{t\in T} A_t$	disjoint union $\bigsqcup_{t\in T} A_t = \bigcup_{t\in T} (A_t \times \{t\})$
Top topological spaces continuous maps	Cartesian product $\prod A_t$ with product topology	disjoint union $\bigsqcup A_t$ $A_t \times \{t\}$ open-and-closed
Top$_\bullet$ pointed topological spaces based maps	Cartesian product $\prod A_t$ with base point $\{\bullet_t\}_{t\in T}$	wedge sum $\bigvee A_t = \left(\bigsqcup A_t\right)/\sim$ with quotient topology
Comp compact spaces continuous maps	Cartesian product $\prod A_t$ with product topology	Stone-Čech compactification of the disjoint union $\beta\left(\bigsqcup A_t\right)$
Gr groups homomorphisms	Cartesian product $\prod A_t$ multiplication componentwise	free product $\coprod A_t$ group of words
Ab abelian groups homomorphisms	Cartesian product $\prod A_t$ addition componentwise	(external) direct sum $\bigoplus A_t$ $x_t = e_t$ for almost all t
Abcomp compact abelian groups continuous homomorphisms	Cartesian product $\prod A_t$ product topology addition componentwise	Bohr compactification of the direct sum $\bigoplus A_t$ with coproduct topology
CRng commutative rings (with units) unit-preserving homomorphisms	Cartesian product $\prod R_t$ addition and multiplication componentwise	tensor product of rings $\bigotimes R_t$
Ban$_1$ Banach spaces linear operators $\|\|T\|\| \leq 1$	ℓ^∞-product the set of all $\{x_t\}_{t\in T}$ satisfying $\sup_{t\in T} \|\|x_t\|\| < \infty$	ℓ^1-sum the set of all $\{x_t\}_{t\in T}$ satisfying $\sum_{t\in T} \|\|x_t\|\| < \infty$
C*algcom$_1$ commutative C^*-algebras unit-preserving homom.	ℓ^∞-product the set of all $\{x_t\}_{t\in T}$ satisfying $\sup_{t\in T} \|\|x_t\|\| < \infty$	Injective tensor product $\widehat{\widehat{\bigotimes}} A_t$
Aut Mealy automata $\langle X, S, Y, \delta, \lambda\rangle$ where $\delta : S \times X \to S$ $\lambda : S \times X \to Y$	$\prod A_t, \prod S_t, \prod Y_t$ induced map δ induced map λ	a construction in terms of disjoint sums of finite Cartesian products

Table 1 In the left-hand column there are symbols of categories together with concise descriptions of their objects and morphisms. In the middle column, for each category there is a description of products. Conspicuously, the cartesian product $\prod$ appears in each cell with exception of Banach spaces and commutative C^*-algebras (however this difference disappears when one changes the definition of the carrier of the object taking the closed unit ball instead of the whole vector space). In the right-hand column there are descriptions of respective coproducts. There are six distinct types of them: coproducts related to the disjoint union $\bigsqcup A_t$; free products $\coprod A_t$; coproducts related to directs sums $\bigoplus A_t$ of abelian groups; related to tensor products $\bigotimes A_t$ of rings; ℓ^1-sums; a specific construction for automata.

In the category **Aut** of finite Mealy automata $\langle X, S, Y, \delta, \lambda \rangle$ (where S denotes a set of states, X is an input alphabet, Y is an output alphabet, $\delta : S \times X \to S$ is a transition function, $\lambda : S \times X \to Y$ is an output function and morphisms are triples $\xi\colon X_1 \to X_2$, $\sigma\colon S_1 \to S_2$, $\eta\colon Y_1 \to Y_2$ such that suitable diagrams commute), the product of $\langle X_t, S_t, Y_t, \delta_t, \lambda_t \rangle$ is the triple $\prod X_t$, $\prod X_t$, $\prod X_t$ with the induced maps δ, λ and is related to that in **Set**, [4]. However, coproducts are sophisticated and quite different [25], [23, 2.4.7, 3.3.15].

2.3 Recapitulation of main points

The forgetful functors from each of the categories considered above to **Set** (or to **Set** $\times$ **Set** $\times$ **Set** in case of automata) commute with products and do not commute with coproducts.

A similar asymmetry, albeit in a much milder form, concern equalizers and coequalizers [23, §3.5].

A consequence of the above asymmetry product–coproduct is an analogous asymmetry of limits (called also inverse limits or projective limits) and colimits (direct limits or inductive limits) of diagrams [15, pp. 62–72].

One may distinguish two kinds of categorial duality. One, which may be labeled as *syntactic*, mentioned above, is based on the formal replacing of each morphism $\alpha\beta$ (in a category $\mathcal{E}$) by $\beta\alpha$. The other, which may be labeled as *functional* (and is an anti-equivalence, i.e., equivalence with the opposite category $\mathcal{E}^{\mathrm{op}}$), is based on constructions related to some contravariant hom-functor $\hom_{\mathcal{E}}(-, E_0)$. The first kind is significant in the general theory, whereas the second yields better insight into specific categories, like these discussed here.

3 A philosophical discussion

At this point a philosophical question arises: *What features of Cantorian Mathematics lie behind this asymmetry?*

Clearly, the membership relation: element–set, $x \in X$ is a basic asymmetry. However, this explanation is not adequate here, as the following examples show.

Let **Rel** denote the category of sets and binary relations. Objects are sets, a morphism $R\colon A \to B$ is a triple (R, A, B) where $R \subseteq A \times B$. If $S \subseteq B \times C$ is another such relation, the composite morphism is $(S \circ R, A, C)$, where

$$S \circ R = \{(a, c) \in A \times C \mid \exists_{b \in B} (a, b) \in R \text{ and } (b, c) \in S\}.$$

The empty set is the zero object. The coproduct of a family $\{A_t\}_{t \in T}$ of objects in **Rel** is the disjoint union $A = \bigsqcup_{t \in T} A_t$ with obvious injections $\sigma_t\colon A_t \to A$.

The product is the same disjoint union $A = \bigsqcup A_t$ with morphisms $\pi_t \colon A \to A_t$ defined as the inverse relations $\pi_t = \sigma_t^{-1}$ for $t \in T$. The categorial symmetry of **Rel** suggests itself.

If a set P is partially ordered by a relation $\leq$ and is regarded as a category in which the morphisms $a \to b$ are exactly those pairs (a, b) for which $a \leq b$ and if the greatest lower bound $\inf\{a_t\}_{t \in T}$ of a family exists, then it is the product of it. Analogously, $\sup\{a_t\}_{t \in T}$ is the coproduct. Here again the symmetry is clear.

The last two examples suggest that **the product-coproduct asymmetry** of the categories shown in the table above follows from **the asymmetry of many-to-one relationship in the notion of a function** $f \colon X \to Y$.

Now a next question arises as to why does such **many→one** thinking predominate in Mathematics. Certainly it is deeply rooted in our minds. It is so in early arithmetic as, e.g., in $4 + 3 = 7$ the natural direction is from numbers $4, 3$ to the sum 7. The opposite relation — decomposing a number into summands — is also important but definitely secondary. Most computations lead from given data to a result. Solving an equation appears to be a way backwards. In calculus, functions play a vital role, whereas their multivalued inverse relations are used only occasionally.

In the real life causes precede the effects [19, Ch. 7]. This is implicit in common thinking, manifests in ordinary language, and also shapes mathematical thinking. It is subordinated to the psychological arrow of time which — according to Hawking [7, Ch. 9] — is determined by the thermodynamic arrow of time. It is likely an evolutionary effect in mathematical thought.

In a preliminary search, in the context of discovery, the mathematician's thinking may have no preconceived direction, but systematic reasoning (as in a proof) has a clear direction (the case of backward reasoning, from the consequent to the antecedent, is usually an intentional, conscious reversing of the direction).

The concept of a function has two aspects: dynamic and static. The first, related to change and motion, was implicit in Newton's approach, continued till the 19th century, and still somehow influences the thinking in terms of functional dependence. Also an "input/output machine" approach, with permissible inputs and the corresponding outputs, is dynamic.

The static conception of a function developed slowly from Euler's analytic form to Dedekind's modern, purely logical and completely general notion of a many-to-one mapping from a set to a set [3, Ch. V–VII], [6, pp. 228–232].

By its very nature, set theory is static. In the set-theoretical approach, the dynamic conception of a function is replaced by a static relation, conceived as a set of pairs. Time, which played the role of a distinct variable in the 18th and 19th century, became one of space coordinates in $\mathbb{R}^n$ [16, pp. 123–133]. Moreover, the New Math movements of the 1960s contributed to the attitude that time belongs to physics. On the other hand, even if formally Mathematics expressed in terms of set theory appears static, Cantorian models of physical processes represent a dynamic world.

The author is indebted to Prof. Jiří Rosický for paying attention to some other important aspects of the question considered in the paper. Most of the examples discussed here are of an "algebraic" nature, i.e., they are given by operations whose general form is $TA \to A$ where $T : \mathbf{Set} \to \mathbf{Set}$ is a functor. But there are also structures of "coalgebraic" nature given by $A \to TA$, [24]. In such a case, coproducts are preserved but not products. A typical example are transition systems given by $A \to \mathcal{P}A$, where $\mathcal{P}$ is the power-set functor [10, part on non-deterministic automata]. Thus, the asymmetry considered in this paper applies to the "algebraic" part of Cantorian Mathematics while an opposite asymmetry applies to the "coalgebraic" part of mathematics. The former is predominant in classical mathematics (particularly in applications to physics) while coalgebraic part is mostly stimulated by Computer Science. The opposite one-to-many relation, for example decomposing a number into summands or multi-valued inverse of a function, is considered as being important but secondary in Cantorian Mathematics. While many-to-one is typical to it, one-to-many is typical to coalgebraic mathematical theories. The example of transition systems shows that one-to-many is not always given by some many-to-one. Here, it reflects the non-deterministic nature of a process where one has more ways how to go from a state of the system to another state.

General theory of coalgebras requires category theory. Before the appearance of the latter, mathematicians could deal only with asymmetric Cantorian Mathematics; the coalgebraic part was hidden.

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